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POLARIZED SURFACES OF Δ-GENUS 3

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ABSTRACT. Let X be a smooth, complex, algebraic, projective surface and let L be an ample line bundle on it. Let $\Delta = \Delta(X, L) = c_1(L)^2+2-h^0(L)$ denote the Δ-genus of the pair (X, L). The purpose of this paper is to classify such pairs under the assumption that $\Delta = 3$ and the complete linear system $|L|$ contains a smooth curve. If $d \geq 7$ and $g \geq \Delta$, Fujita has shown that L is very ample and $g = \Delta$. If $d \geq 7$ and $g < \Delta = 3$, then $g = 2$ and those pairs have been studied by Fujita and Beltrametti, Lanteri, and Palleschi. To study the remaining cases we have examined the two possibilities of $L + tK$ being nef or not, for $t = 1, 2$. In the cases in which $L + 2K$ is nef it turned out to be very useful to iterate the adjunction mapping for ample line bundles as it was done by Biancofiore and Livorni in the very ample case. If $g > \Delta$ there are still open cases to solve in which completely different methods are needed.

INTRODUCTION

By a polarized variety we mean a pair (X, L) with X a smooth, complex, projective variety and L an ample line bundle on it. For such a pair Fujita [F_1] has introduced the concept of Δ-genus which is defined by $\Delta = \Delta(X, L) = n + d - h^0(L)$, where $n = \dim X$ and $d = L^n$. In the past few years the structure of polarized varieties has been studied extensively by several authors and most of the work has been done by Fujita. While the structure of such a pair with $\Delta = 0$ is completely known in the case $\Delta = 1, 2$ there are still hard cases to be understood (see [F_4, F_8, F_9]).

The purpose of this paper is to give a contribution to the classification of polarized surfaces with $\Delta = 3$, under the assumption that the complete linear system $|L|$ contains a smooth curve. The results we find are too complicated to be described here, thus we have summarized them in the tables at the end of the paper. We have been mainly inspired by Fujita's method in [F_6] to classify surfaces with sectional genus two. In some cases, precisely when $2K + L$ is nef, it turned out to be very useful to iterate the adjunction mapping for ample line bundles as was done in [BL_1, BL_2] in the very ample case. Actually the combination of the two methods can be used to give a more detailed description of such pairs but for this it is necessary to do a case by case analysis of the many

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possibilities occurring for the values of $g(L+tK)$ for fixed d and $g(L)$. In this paper we have used the second argument only for pairs with $2K+L$ nef, $g(K+L)\leq 2$ or $g(K+L)=3$, and $\Delta(X, K+L)\leq 3$.

0. Background material

(0.0) Let X be a smooth surface and L an ample line bundle on X. We use the following notation:

$K=$ the canonical bundle of X,
$g=g(L)$, the sectional genus of (X, L),
$d=L^2$, the degree of the line bundle L,
$q=q(X)$, the irregularity of X,
$p_g=p_g(X)$, the geometric genus of X,
$\Delta=\Delta(X, L)=2+d-h^0(L)$, the Δ-genus of the pair (X, L), and
$\kappa(X)=$ the Kodaira dimension of X.

(0.1) **Definition.** According to Fujita [F_6, (4.8)], a polarized surface which is not a minimal model is said to be *half-minimal* if $2K+L$ is nef.

If (X, L) is half-minimal then $L\cdot E\geq 2$ for any exceptional curve E on X. We let s be the number of exceptional curves E such that $L\cdot E\geq 2$.

(0.2) *Remark.* Let X be a surface whose canonical bundle is not nef. If (X, L) is not half-minimal and not minimal, then there exists an exceptional curve E on X which is a line relative to L. Let (X', L') be the pair obtained from (X, L) after we blow down E, and let $\pi\colon X\to X'$ be the blow down map. Let $\pi^*L'=L+E$. Note that $\pi^*(K'+L')=K+L$, where K' denotes the canonical bundle of X'. When there is no danger of confusion we write L' for π^*L' and $K'+L'$ for $\pi^*(K'+L')$. We also have $g(L')=g$. After we blow down all the exceptional curves which are lines relative to L we get another pair (X_0, L_0) such that one of the following holds:

(1) K_0 is nef.
(2) (X_0, L_0) is a $\mathbf{P}^1$-bundle over a smooth curve.
(3) (X_0, L_0) is half-minimal.
(4) $(X_0, L_0)=(\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(\delta))$.

Note that this last possibility did not occur in [F_6, (4.13)]. Note also that the cases (1) through (4) have to be examined at each step. We denote by r the number of exceptional curves E such that $L\cdot E=1$.

(0.3) **Definition.** Unless (X_0, L_0) is a minimal model, the pair (X_0, L_0) is called a *relatively half-minimal model* of (X, L).

(0.4) *Remark.* If $K+L$ is nef and the morphism associated to $|(K+L)|$ has a two-dimensional image, then the pair (X_0, L_0) is the minimal reduction defined by Sommese in [S_2].

(0.5) *Remark.* Note that $g=0, 1$ implies $\Delta\leq 2$ [F_4, F_5, La], under the assumption that the complete linear system $|L|$ contains a smooth curve.

(0.6) **Lemma.** *Let* (X, L) *be as in* (0.0). *If* $g \leq \Delta - 1$, *then* $q \neq 0$.

Proof. Assume that $q = 0$. Then from the long cohomology sequence associated to the short exact sequence

$$0 \to \vartheta_X \to L \to L_C \to 0, \tag{0.6.1}$$

we see that the restriction map $\Gamma(X, L) \to \Gamma(C, L_C)$ is onto. Hence by [F_2, (1.5)], $\Delta(X, L) = \Delta(C, L_C)$. On the other hand, by Riemann-Roch, $\Delta(C, L_C) = g - h^1(L_C) \leq g \leq \Delta - 1$, which gives a contradiction. □

For the convenience of the reader we state the following theorem which is used throughout the paper.

(0.7) **Theorem.** *Let* X *be a smooth surface and let* L *be an ample line bundle on* X. *Then* $K + L$ *is nef unless*

(0.7.1) $(X, L) = (\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(e))$, $e = 1, 2$,
(0.7.2) X *is a scroll over a curve of genus* g,
(0.7.3) X *is a quadric in* $\mathbf{P}^3$ *and* $L = \vartheta_X(1)$.

For a proof see [F_5, LP, S_3].

(0.8) Let X be a $\mathbf{P}^1$-bundle over a smooth curve B, i.e. a geometrically ruled surface, let L be an ample line bundle over X, and let $L \equiv aC_0 + bf$ for some $a, b \in \mathbf{Z}$. Then we have the following system:

$$\begin{cases} 2g - 2 = -a^2e + ae + 2ab - 2b - 2a + 2aq, \\ L^2 = -a^2e + 2ab. \end{cases} \tag{0.8.1}$$

For the notation about ruled surfaces see [H].

1. The case $d \geq 7$

(1.0) Throughout the paper we let X be a smooth surface, L be an ample line bundle on it such that there exists a smooth element C in the linear system $|L|$, and $\Delta = 3$.

(1.0.1) *Remark.* Note that since there exists a smooth $C \in |L|$ then $\dim Bs|L| \leq 0$ if $d \geq 3$.

(1.1) **Theorem.** *Let* X *be a* $\mathbf{P}^1$*-bundle over a smooth curve* B *and let* $L \equiv aC_0 + bf$ *be an ample line bundle on* X, *with* $a \geq 2$. *Assume that* $g = \Delta(X, L) = 3$. *Then* $d = 16$, $(X_0, L_0) = (\mathbf{F}_e, 2C_0 + (e + 4)f)$, $e \leq 3$, *or* $(X_0, L_0) = (\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(4))$.

Proof. By (0.7) we see that $K + L$ is nef unless (X, L) is a scroll over a curve of genus 3. Hence if (X, L) is not a scroll we have $(K + L)^2 \geq 0$. Also

$$(K + L)^2 \leq 8(1 - q) + 2(2g - 2) - d. \tag{1.1.0}$$

Thus $8q \leq 16 - d$, i.e. $q \leq 1$. If $d \geq 7$, by [F_2, (3.6)] L is very ample. Then by [BL_1, BL_2] it follows that (X, L) has a reduction (X_0, L_0) as above. If

$d < 7$ then by (0.8.1) we obtain the following:

(1.1.1) $1 - 3a^{-1} - a + aq = 0$ if $d = 6$,
(1.1.2) $1 - 5a^{-1} - 2a + 2aq = 0$ if $d = 5$,
(1.1.3) $2a^{-1} - a + aq = 0$ if $d = 4$,
(1.1.4) $1 - 3a^{-1} - 2a + 2aq = 0$ if $d = 3$,
(1.1.5) $-1 - a^{-1} - a + aq = 0$ if $d = 2$.

Note that (1.1.3), (1.1.4), and (1.1.5) do not have any integral solutions for $q = 0, 1$. As for (1.1.1) and (1.1.2) we get a contradiction for $q = 0$ while for $q = 1$ we have $a = 3$, $b = \frac{1}{2}(3e + 2)$ and $a = 5$, $b = \frac{1}{2}(5e + 1)$ respectively. By [H, 2.12 and Example 2.5] and by the ampleness of L we see that either $L \equiv 3C_0 + f$, $e = 0$ if $d = 6$, or $L \equiv 5C_0 - 2f$, $e = -1$ if $d = 5$. By [Li_2, (0.5.7)] we have $h^1(L) = 0$ and $h^0(L) = 4$ or 3 according as $d = 6$ or 5. Hence $\Delta(X, L) \neq 3$, which contradicts our hypothesis. □

(1.2) **Theorem.** *Let X, L, and Δ be as in* (1.0). *Assume that* $L^2 = d \geq 7$. *Then either*

(1.2.1) $g = 2$, (X, L) *is a scroll over a smooth curve of genus two, or*
(1.2.2) $g = 3$, (X, L) *is a scroll over a smooth curve of genus three, or*
(1.2.3) $g = 3$, $d_0 = 16$, $r = 0, \ldots, 9$, $(X_0, L_0) = (\mathbf{F}_e, 2C_0 + (e + 4)f)$, $e \leq 3$, *or*
(1.2.4) $g = 3$, $d_0 = 16$, $r = 0, \ldots, 9$, $(X_0, L_0) = (\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(4))$, *or*
(1.2.5) $g = 3$, $d_0 = 8$, $r = 0, 1$, $(X_0, L_0) = (\mathbf{P}^2_7, \vartheta_{\mathbf{P}^2}(6) - 2\sum_{i=1}^{7} E_i)$.

Proof. The proof will be done according to $g \geq \Delta$ or $g < \Delta$. We will first look at the case $g \geq \Delta$. From [F_2, (3.6)] it follows that L is very ample and $\Delta = g = 3$. From the adjunction formula we have

(1.2.6) $$K_X \cdot L = 2g - 2 - d < 0.$$

Thus $\kappa(X) = -\infty$. Either we have (1.2.2) or from [BL_1, BL_2] we get the cases (1.2.3), (1.2.4), and (1.2.5). Now assume that $g < \Delta$, i.e. $g \leq 2$. By (0.5), $g = 2$. Moreover $K \cdot L = 2g - 2 - d \leq -5$, which implies that $\kappa(X) = -\infty$. Thus either $X = \mathbf{P}^2$ or X is ruled. Note that $X \neq \mathbf{P}^2$ since $g = 2$. By (0.7) we see that $K + L$ is nef unless (X, L) is as in (1.2.1). Hence if (X, L) is not as in (1.2.1) then by (1.1.0) we have $q = 0$ which contradicts (0.6). Thus we see that (1.2.1) is the only possible case. □

2. The case $d = 6$

Let $g \leq \Delta = 3$. By (0.5), $g = 2, 3$. From (1.2.6) it follows that K is not nef and $\kappa(X) = -\infty$. Moreover it is easy to see that $X \neq \mathbf{P}^2$ for $g = 2, 3$. Hence X is ruled.

(2.1) $g = 2$. By (0.7) $K + L$ is nef unless (X, L) is a scroll over a curve B of genus two. In the case $K + L$ nef, by (1.1.0) $q = 0$ which contradicts (0.6).

Hence (X, L) is a scroll. Let $L \equiv C_0 + bf$. We have

$$6 = L^2 = -e + 2b. \tag{2.1.1}$$

By [H, 2.12 and Example 2.5] we get that $e \geq 0$ if $\mathscr{E}$ is decomposable and $-q \leq e \leq 2q-2$ if $\mathscr{E}$ is indecomposable, where $\mathscr{E}$ is a locally free sheaf of rank two on B such that $X = \mathbf{P}_B(\mathscr{E})$. Using the ampleness of L we have $L \equiv C_0 + \frac{1}{2}(6+e)f$, with $e = -2, 0, 2, 4$. We now use [Li$_2$, (0.5.7)] along with $\Delta = 3$ and $q = 2$ to conclude that $e = 2, 4$.

(2.2) $g = 3$. By (0.7) we see that $K + L$ is nef unless (X, L) is a scroll over a curve of genus 3. In the latter case, as in the proof of (2.1), we conclude that $-3 \leq e \leq 4$. Thus we have $L \equiv C_0 + \frac{1}{2}(6+e)f$ with $e = -2, 0, 2, 4$. Note that since $d = 6 = 2\Delta$, by [F$_2$, (3.6)] we get that L is spanned. Hence L_{C_0} is also spanned. Thus $\deg L_{C_0} \geq 2(e+3)$ which implies $e \neq 4$.

From now on we assume $K+L$ nef. Hence by (1.1.0) we have $8q \leq 10$, i.e. $q \leq 1$. Moreover by (1.1) X is not a $\mathbf{P}^1$-bundle. Thus there exists a smooth rational curve E on X such that $E^2 = -1$.

(2.2.1) *Claim.* $2K + L$ is not nef.

Proof of the claim. Assume otherwise, i.e. $2K+L$ is nef. Then $0 \leq (2K+L)^2 = 4K^2 + 4K \cdot L + L \cdot L = 4K^2 - 2$ which gives $K^2 \geq 1$. If $q = 1$ we have $K^2 \leq 8(1-q) = 0$, hence a contradiction. If $q = 0$ then since $(3K+L) \cdot L = 0$ by the Hodge Index Theorem we have $0 \geq (3K+L)^2 = 9K^2 - 6$ which gives $K^2 \leq 0$, again a contradiction. Hence $2K + L$ is not nef and (X, L) is not a $\mathbf{P}^1$-bundle.

By (0.2) there is an exceptional curve E on X which is a line relative to L, i.e. $L \cdot E = 1$. Let (X', L') be the pair obtained from (X, L) after we blow down E. We have the following possibilities:

(a) K' is nef.
(b) X' is a $\mathbf{P}^1$-bundle.
(c) X' is such that $2K' + L'$ is nef.
(d) There exists an exceptional curve E' on X' with $L' \cdot E' = 1$.
(e) $(X', L') = (\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(\delta))$.

Note that (a) does not happen since $\kappa(X) = -\infty$. If X' is as in (b) then substituting $d' = L'^2 = 7$, $g' = g(L') = 3$, and $q = 0, 1$ in (0.8.1) we have $2a^2 - 3a + 7 = 0$ if $q = 0$ and $3a - 7 = 0$ if $q = 1$ which are both impossible since $a \in \mathbf{Z}$. Hence case (b) cannot occur either. If X' is as in (c) then $0 \leq (2K' + L')^2 = 4K'^2 + 4K' \cdot L' + L' \cdot L' = 4K'^2 - 5$, i.e. $K'^2 \geq 2$. Also by the Hodge Index Theorem we have $(7K'^3 + L')^2 \cdot L'^2 \leq ((7K'^3 + L') \cdot L')^2 = 0$. Hence $K'^2 \leq 1$ which contradicts the above inequality. Thus case (c) cannot occur either. Since $d' = 7$ also case (e) cannot occur. If X' is as in (d) i.e.

$r \geq 2$ then we blow down all the exceptional curves which are lines relative to L. Either $(X_0, L_0) = (\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(\delta))$ or (X_0, L_0) is as in (b) or in (c). In the first case we have $\delta = 4$, $r = 10$ (this is the well-known Bordiga surface). Now assume that (X_0, L_0) is not a $\mathbf{P}^1$-bundle. Hence $0 \leq (2K_0 + L_0) \cdot L_0 = 4g(L_0) - 4 - d_0 = 8 - d_0$, i.e. $d_0 \leq 8$, where K_0 is the canonical bundle of X_0 and $d_0 = L_0^2$. On the other hand $d_0 = d + r$. Thus $r = 2$.

Since $2K_0 + L_0$ is nef and $(2K_0 + L_0) \cdot L_0 = 0$ by the Hodge Index Theorem we have $2K_0 + L_0 \sim \vartheta_{X_0}$. Thus (X_0, L_0) is a Del Pezzo surface and it is the blow up of $\mathbf{P}^2$ at seven points, and $L_0 = \vartheta_{\mathbf{P}^2}(6) - 2\sum_{i=1}^{7} E_i$. If at any step before the final one we get a $\mathbf{P}^1$-bundle then either $r = 2$, $q = 1$, $L \equiv 2C_0 + (e+2)f$, $e = -1, 0, 1$ or $r = 10$, $q = 0$, $L \equiv 2C_0 + (e+4)f$, $e \leq 3$.

We now look at the case $g > \Delta = 3$. By [F_3, (1.4), (1.10) and Table I], if $g = 4$ either X is a K-3 surface or X is of type $(\Sigma(2,1)^+_{2-1})$, $(\Sigma(2,1)^+_{12})$, $(\Sigma(2,1)^0_5)$. While if $g \geq 5$ then either X is of type $(\Sigma(2, 1)^+_{ab})$ or $(\Sigma(2, 1)^0_b)$. We refer to [F_3] for details about surfaces of type (Σ). The list of polarized surfaces with $\Delta = 3$ and $d = 6$ can be found in Table II. (Tables I–VI can be found at the end of §6.)

3. The case $d = 5$

Let $g \leq \Delta = 3$. By (0.5), $g = 2, 3$. From (1.2.6) we have that K is not nef and X is ruled.

(3.1) $g = 2$. By (0.7) $K + L$ is nef unless (X, L) is a scroll over a curve B of genus two. In the case $K + L$ nef, by (1.1.0) $q = 0$ which contradicts (0.6). Hence (X, L) is a scroll. Then as in (2.1) we see that $L \equiv C_0 + \frac{1}{2}(5+e)f$, with $e = -1, 1, 3$.

(3.2) $g = 3$. By (0.7) $K + L$ is nef unless (X, L) is a scroll over a curve of genus three. Assume that (X, L) is a scroll. Then as in (2.2) we see that $L \equiv C_0 + \frac{1}{2}(5+e)f$, with $e = -3, -1, 1, 3$.

From now on we assume that $K + L$ is nef. As we have already seen, K is not nef, $X \neq \mathbf{P}^2$, and by (1.1) X is not a $\mathbf{P}^1$-bundle. Thus there exists a smooth rational curve E on X such that $E^2 = -1$.

We divide the study into two cases according to $2K + L$ being nef or not nef. If $2K + L$ is nef then

(3.2.1) $$0 \leq (2K + L)^2 = 4K^2 + 4K \cdot L + L \cdot L.$$

Thus $4K^2 + 1 \geq 0$, i.e. $K^2 \geq 0$. Let X' be the surface obtained by contracting the exceptional curve E on X. Hence $K^2 = K'^2 - 1 \leq 8(1-q) - 1$. If $q = 1$ we have $K^2 \leq -1$, a contradiction. If $q = 0$ then since $(5K + L) \cdot L = 0$ by the Hodge Index Theorem we have $0 \geq (5K + L)^2 = 25K^2 - 5$. Hence $K^2 = 0$.

Therefore X is obtained by blowing up either eight points on $\mathbf{F}_e$ or nine points on $\mathbf{P}^2$. By [LP, Theorem 2.5] it is easy to see that $K+L$ is ample. Moreover $(K+L)^2 = 3$ and $g(K+L) = 2$. Thus by the classification of surfaces of sectional genus two we have

$$(X, L) = \left(\mathbf{P}_9^2, \vartheta_{\mathbf{P}^2}(9) - 3\sum_{i=1}^{8} E_i - 2E_9\right).$$

If $2K+L$ is not nef then by (0.2) there exists an exceptional line E. Let (X', L') be the pair obtained from (X, L) after we blow down E. Note that $d' = 6$, $g' = g = 3$, and $K' \cdot L' = -2$. Since $X' \neq \mathbf{P}^2$ the pair (X', L') is either as in (b) or there exists a smooth rational curve E on X' such that $E^2 = -1$.

If (X', L') is as in (b) then $q = 1$, $(X', L') = (\mathbf{P}^1\text{-bundle}, 3C_0 + f)$, and $e = 0$.

It is easy to check that $2K' + L'$ is not nef.

If (X', L') has a smooth exceptional curve E then we have to look at Table II (the case $g = 3$ and $2K+L$ not nef), since the argument used does not depend on the assumption $\Delta = 3$.

We now look at the case $g > \Delta = 3$. By (1.0.1) $\dim Bs|L| \leq 0$. We treat only the case $\dim Bs|L| < 0$, i.e. L is spanned. Let Φ denote the morphism associated to the linear system $|L|$. We have

$$5 = \deg \Phi \cdot \deg \Phi(X) \geq \deg \Phi(h^0(L) - 2).$$

Thus Φ is generically one-to-one. Hence by Castelnuovo's formula we get that $g \leq 6$.

(3.3) $g = 4$. By the adjunction formula we have $K \cdot L = 1$. Hence by the Hodge Index Theorem $K^2 \leq 0$. We also know by (0.7) that $K+L$ is nef unless (X, L) is a scroll over a curve of genus 4. Assume that (X, L) is a scroll. Then as in (2.1) we see that $L \equiv C_0 + \frac{1}{2}(5+e)f$, with $e = -3, -1, 1, 3$.

From now on we assume that $K+L$ is nef. We will consider separately the case K nef and K not nef.

(3.3.1) K *nef.* Since K is nef, $K^2 \geq 0$. But $K^2 \leq 0$ by the Hodge Index Theorem. Thus $K^2 = 0$. Moreover since K is nef then X has to be a minimal model. Note that $\kappa(X) \neq 0$ since $K \cdot L = 1$. So we can conclude that $\kappa(X) = 1$.

(3.3.2) K *not nef.* Note that $X \neq \mathbf{P}^2$. Moreover by (0.8.1) X is not a $\mathbf{P}^1$-bundle. Thus there exists a smooth rational curve E on X such that $E^2 = -1$.

We first look at the case $2K+L$ nef. If $2K+L$ is nef then by (3.2.1) $0 \leq 4K^2 + 9$. Hence $K^2 \geq -2$ and thus $K^2 = -2, -1, 0$. As in [F_6, (4.9)] we conclude that X is ruled and by (1.1.0) $q \leq 1$. Since X is not a minimal model and $K \cdot L = 1$ then $K^2 < 0$ by [I, (1.4)].

(3.3.3) Let $K^2 = -1$ and $q = 1$. Then X is obtained by blowing up one point on a minimal model. By [F_7, (5.15)] $m = 2, 3$. Let $L \equiv aC_0 + bf - mE$, where $L \cdot E = m \geq 2$, and E is an exceptional curve on X. Thus

$$5 = L^2 = -a^2e + 2ab - m^2 \quad \text{and} \quad 1 = K \cdot L = ae - 2b + m.$$

Hence $2b = m + ae - 1$ and $m^2 - am + a + 5 = 0$. From the latter equation we have $(2m - a)^2 = a^2 - 4a - 12$. Completing the squares we have $(a-2)^2 - (2m-a)^2 = 24$, i.e. $(a - m - 1)(m - 1) = 6$. Thus either $a = 9$ and $m = 2$, or $a = 7$ and $m = 3$. Note that $e \geq -1$. Thus for $e = -1$, from $2b = m + ae - 1$ we have $b = -3$ if $a = 7$ and $m = 2$, and $b = -2$ if $a = 6$ and $m = 3$. If $e \geq 0$ then by the ampleness of L we have $e = 0$. Hence for $e = 0$ we have $b = 1$ if $a = 6$ and $m = 3$. Thus $L \equiv 7C_0 - 3f - 2E$, $e = -1$, or $L \equiv 6C_0 + (3e+1)f - 3E$, $e = -1, 0$.

Let $K^2 = -1$ and $q = 0$. Then X is obtained by blowing up either nine points on $\mathbf{F}_e$ or ten points on $\mathbf{P}^2$. In the first case by [F_7, (5.15)] $m = 2, 3$.

(3.3.4) Let $K^2 = -2$. If $q = 1$ then X is obtained by blowing up two points on a $\mathbf{P}^1$-bundle X'. Let Q_i denote the free base for $\operatorname{Num} X$ defined in [BL_3, (1.0)]. Then $L \equiv aC_0 + bf - m_1Q_1 - m_2Q_2$. Hence $1 = K \cdot L = ae - 2b + m_1 + m_2$ which gives $2b - ae = m_1 + m_2 - 1$. Substituting this in $5 = L^2 = -a^2e + 2ab - m_1^2 - m_2^2$ we have

$$6 = (m_1 - 1)(a - m_1 - 1) + m_2(a - m_2). \tag{3.3.5}$$

By (3.3.5) and by [F_7, (5.15)] we get a contradiction.

Let $K^2 = -2$ and $q = 0$. Then X is obtained by blowing up either ten points on $\mathbf{F}_e$ or eleven points on $\mathbf{P}^2$. By [LP, Theorem 2.5] it is easy to see that $K+L$ is ample. Moreover $(K+L)^2 = 5$ and $g(K+L) = 3$, $h^0(K+L) = 4$. Thus $\Delta(X, K+L) = 3$. By our previous classification we see that the first case does not occur.

(3.3.6) We now look at the case $2K + L$ not nef. By (0.2) there is an exceptional line E on X. We blow down all such curves. We let (X_0, L_0) be a relatively half-minimal model of (X, L). Note that $(X_0, L_0) \neq (\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(\delta))$. If $2K_0 + L_0$ is nef then $0 \leq (2K_0 + L_0) \cdot L_0 = 4g(L_0) - 4 - d_0 = 12 - d_0$, i.e. $d_0 \leq 12$, where K_0 is the canonical bundle of X_0 and $d_0 = L_0^2$. On the other hand, $d_0 = d + r$, where r is the number of exceptional curves which are lines relative to L. Thus $r \leq 7$.

Let $r = 1$. Then $d_0 = 6$ and $K_0 \cdot L_0 = 0$. Thus either $K_0 \sim \vartheta_{X_0}$ or K_0 is not nef. In the first case $\kappa(X) = 0$ and since $K_0^2 = 0$ and $K_0 \cdot L_0 = 0$ it follows that X_0 is a minimal model. Moreover $h^i(L) = 0$ for $i > 0$ and hence

$$\chi(L_0) = h^0(L_0) = \chi(\vartheta_{X_0}) + \tfrac{1}{2}L_0 \cdot (L_0 - K_0) = \chi(\vartheta_{X_0}) + 2.$$

As in (3.3.1) we conclude that X_0 is either an Enriques surface or a K-3 surface.

If $\kappa(X) = -\infty$ then by (1.1.0) $q \leq 1$. Since $2K_0 + L_0$ is nef by (3.2.1) we have $0 \leq 4K_0^2 + 6$. Hence $K_0^2 \geq -1$. On the other hand, by the Hodge Index Theorem we have $K_0^2 \leq 0$ and thus $K_0^2 = -1, 0$.

If $q = 1$ and $K_0^2 = 0$ then X_0 is a $\mathbf{P}^1$-bundle. From (0.8.1) we get a contradiction. If $q = 1$ and $K_0^2 = -1$ then $m = 2$, [F$_7$, (5.15)]. Now reasoning as in (3.3.3) we get $L \equiv 5C_0 + f - 3E_1 - E_2$ and $e = 0$.

(3.3.7) If $q = 0$ and $K_0^2 = 0$ then X_0 is not a minimal model. Thus there exists an exceptional curve E on X_0 such that $L_0 \cdot E = m \geq 2$. Let X_1 be the surface obtained from X_0 after we blow down E. Thus $d_1 = 6 + m^2$, $K_1^2 = 1$, and $K_1 \cdot L_1 = -m$. By the Hodge Index Theorem $K_1^2 \cdot L_1^2 \leq (K_1 \cdot L_1)^2$ which gives a contradiction by plugging in the respective values. If $q = 0$ and $K_0^2 = -1$ by [F$_7$, (5.15)] $m = 2$. Moreover X is obtained by blowing up either nine points on $\mathbf{F}_e$ or ten points on $\mathbf{P}^2$. By [LP, Theorem 2.5] it is easy to see that $K_0 + L_0$ is ample. Moreover $(K_0 + L_0)^2 = 5$ and $g(K_0 + L_0) = 3$, $h^0(K_0 + L_0) = 4$. Thus $\Delta(X_0, K_0 + L_0) = 3$. By our previous classification we see that the first case does not occur. Let $r \geq 2$. Since $2K_0 + L_0$ is nef by (3.2.1) we have $0 \leq 4(K^2 + r) + 4(1 - r) + 5 + r = 4K^2 + 9 + r$. Thus

$$-(9 + r)/4 \leq K^2 \leq 0. \qquad (3.3.8)$$

Note that since $r \geq 2$ we have $K_0 \cdot L_0 = 1 - r < 0$. Hence X_0 is ruled. Moreover by (1.1.0) $q \leq 1$.

Let $q = 1$. Then $K_0^2 \leq 8(1 - q) = 0$. Moreover $K_0^2 = K^2 + r$. Thus by (3.3.8) we have $r \leq 3$ which gives $K_0^2 = 0$ and hence X_0 is a $\mathbf{P}^1$-bundle. By (0.8.1) if $r = 2, 3$ we have $a = 7, 4$ respectively. It is easy to see that if $r = 2$ then $L_0 \equiv 7C_0 - 6f$ and $e = -1$, and if $r = 3$ then $L_0 \equiv 4C_0 + f$ and $e = 0$.

Let $q = 0$. By the Hodge Index Theorem $K_0^2 \leq (1 - r)^2/(5 + r)$. If $r = 5$ then $K_0^2 \leq 1$. On the other hand by (3.3.8) we have $2 \leq K_0^2 \leq 5$ which gives a contradiction. If $r = 6$ then $K_0^2 \leq 2$. Again by (3.3.8) we have a contradiction. Thus either $r = 7$ or $2 \leq r \leq 4$. For $r = 4, 7$ we can determine the adjoint spectrum of (X, L) (see [F$_7$] for such a definition).

Let $r = 4$. Then $d_0 = 9$, $K_0 \cdot L_0 = -3$, and $K_0^2 \leq 1$. Also by (3.3.8) $1 \leq K_0^2 \leq 4$. Thus $K_0^2 = 1$. As in (3.3.7) we have $K_1^2 \cdot L_1^2 \leq (K_1 \cdot L_1)^2$, where $L_1^2 = d_1 = 9 + m^2$, $K_1 \cdot L_1 = -3 - m$, and $K_1^2 = 2$. Thus $2(9 + m^2) \leq (-3 - m)^2$, i.e. $m = 3$. We continue this process until we get a minimal model. Note that at each step we obtain $m = 3$. Hence the adjoint spectrum is either $(m_1, \ldots, m_4, m_5, \ldots, m_{12}) = (1, \ldots, 1, 3, \ldots, 3)$ or $(m_1, \ldots, m_4, m_5, \ldots, m_{13}) = (1, \ldots, 1, 3, \ldots, 3)$.

Let $r = 7$. Then $d_0 = 12$, $K_0 \cdot L_0 = -6$, and $K_0^2 \leq 3$. Also by (3.3.8) $3 \leq K_0^2 \leq 7$. Thus $K_0^2 = 3$. Hence X_0 is not a minimal model. We compute the adjoint spectrum of (X, L) as in the case $r = 4$. Thus we get either

$$(m_1, \ldots, m_7, m_8, \ldots, m_{12}) = (1, \ldots, 1, 2, \ldots, 2)$$

or

$$(m_1, \dots, m_7, m_8, \dots, m_{13}) = (1, \dots, 1, 2, \dots, 2).$$

By [F_7, (5.13), (5.15)] we have that if $r = 2$ then $K_0^2 = 0$, $m = 2, 3$ and $s \le 8$, while if $r = 3$ then $K_0^2 = 0$, $m = 2$ and $s \le 8$.

(3.4) $g = 5$. By the adjunction formula we have $K \cdot L = 3$. Hence by the Hodge Index Theorem $K^2 \le 1$. We also know by (0.7) that $K + L$ is nef unless (X, L) is a scroll over a curve of genus 5. Assume that (X, L) is a scroll. Then as in (2.1) we see that $L \equiv C_0 + \frac{1}{2}(5 + e)f$, with $e = -5, -3, -1, 1, 3$.

From now on we assume that $K + L$ is nef.

(3.4.1) K *nef.* Since K is nef, $K^2 \ge 0$ and X has to be a minimal model. But $K^2 \le 1$ by the Hodge Index Theorem. Thus $K^2 = 0, 1$. Since $K \cdot L = 3$ we see easily that $\kappa(X) = 1$ if $K^2 = 0$ and $\kappa(X) = 2$ if $K^2 = 1$.

(3.4.2) K *not nef.* Note that $X \ne \mathbf{P}^2$. Moreover by (0.8.1) X is not a $\mathbf{P}^1$-bundle. Thus there exists a smooth rational curve E on X such that $E^2 = -1$.

We first look at the case $2K + L$ nef. If $2K + L$ is nef then by (3.2.1) $0 \le 4K^2 + 17$. Hence $K^2 \ge -4$. Thus $-4 \le K^2 \le 1$. As in [F_6, (4.9)] we conclude that either X is ruled and $q = 0, 1$ or $\kappa(X) \ge 0$. If $\kappa(X) \ge 0$, then, since $K \cdot L = 3$, $2K + L$ is nef, and $2 \le m_i \le 3$, analyzing the various value of m_i and applying the Hodge Index Theorem to the pair (X', L') we have the following two possibilities:

(1) $m = 3$, $\kappa(X) = 0$, $K^2 = -1$,
(2) $m = 2$, $\kappa(X) = 1$, $K^2 = -1$.

Assume now that X is ruled. By [I, (1.4)] since $K \cdot L = 3$ then $K^2 \le -1$.

Let $K^2 = -1$. If $q = 1$ then as in (3.3.3) we have

$$5 = L^2 = -a^2 e + 2ab - m^2 \quad \text{and} \quad 3 = K \cdot L = ae - 2b + m.$$

Combining these two equalities we get $m^2 - am + 3a + 5 = 0$ or equivalently $(a - m - 3)(m - 3) = 14$. Thus either $a = 21$ and $m = 4, 17$, or $a = 15$ and $m = 10$. Since by [F_7, (5.15)] $m \le 6$ we conclude that the only possibility for L is $L \equiv 21C_0 - 10f - 4E$ and $e = -1$.

Let $K^2 = -1$ and $q = 0$. Then X is obtained by blowing up either nine points on $\mathbf{F}_e$ or ten points on $\mathbf{P}^2$. As before we see that $m \le 6$.

Let $K^2 = -2$. If $q = 1$ then as in (3.3.4) and by [F_7, (5.15)] we get a contradiction.

Let $q = 0$. Then X is obtained by blowing up either ten points on $\mathbf{F}_e$ or eleven points on $\mathbf{P}^2$. By [F_7, (1.15)] $m = 2, 3$.

If $K^2 = -3, -4$ then $m = 2$ and $q = 0$.

If $2K + L$ is not nef then, as in (3.3.6), we let (X_0, L_0) be a relatively half-minimal model of (X, L). If $2K_0 + L_0$ is nef we have $d_0 \le 16$. On the other hand $d_0 = d + r$, where r is the number of exceptional curves which are lines relative to L. Thus $r \le 11$. Since $2K_0 + L_0$ is nef by (3.2.1) we have

$0 \leq 4(K^2 + r) + 4(3 - r) + 5 + r = 4K^2 + r + 17$. We also know that $K^2 \leq 1$. Thus

(3.4.3) $$-(r + 17)/4 \leq K^2 \leq 1.$$

By the Hodge Index Theorem applied to K_0 and L_0 we have

$$K_0^2 \leq (3 - r)^2/(5 + r).$$

Moreover if $r \geq 4$ then $K_0 \cdot L_0 = 3 - r < 0$. Hence X_0 is ruled, and $q \leq 1$. The same analysis as in case $g = 4$ can be done for each value of r.

(3.5) $g = 6$. By the adjunction formula we have $K \cdot L = 5$. Hence by the Hodge Index Theorem $K^2 \leq 5$. We also know by (0.7) that $K + L$ is nef unless (X, L) is a scroll over a curve of genus 6. Let (X, L) be a scroll. Then as in (2.1) we see that $L \equiv C_0 + \frac{1}{2}(5 + e)f$, with $e = 5, -3, -1, 1, 3$.

From now on we assume that $K + L$ is nef.

(3.5.1) K nef. Since K is nef, $K^2 \geq 0$ and X has to be a minimal model. But $K^2 \leq 5$ by the Hodge Index Theorem. Thus $0 \leq K^2 \leq 5$. Since $K \cdot L = 5$ we see easily that $\kappa(X) = 2$ if $1 \leq K^2 \leq 5$ and $\kappa(X) = 1$ if $K^2 = 0$.

(3.5.2) K *not nef.* Note that $X \neq \mathbf{P}^2$. Moreover by (0.8.1) X is not a $\mathbf{P}^1$-bundle. Thus there exists a smooth rational curve E on X such that $E^2 = -1$.

We first look at the case $2K + L$ nef. If $2K + L$ is nef then by (3.2.1) $0 \leq 4K^2 + 25$. Hence $K^2 \geq -6$. Applying the Hodge Index Theorem to the pair (X', L') it follows that $-6 \leq K^2 \leq 0$. As in [F_6, (4.9)] we conclude that either X is ruled and $q = 0, 1$ or $\kappa(X) \geq 0$. If $\kappa(X) \geq 0$, then, since $K \cdot L = 5$, $2K + L$ is nef, and $2 \leq m_i \leq 5$, analyzing the various value of m_i and applying the Hodge Index Theorem to the pair (X', L') we have the following possibilities:

(1) $m = 5$, $\kappa(X) = 0$, $K^2 = -1$,
(2) $m = 4$, $\kappa(X) = 1$, $K^2 = -1$,
(3) $m = 3$, $\kappa(X) = 1$, $K^2 = -1$,
(4) $m_1 = 3$, $m_2 = 2$, $\kappa(X) = 0$, $K^2 = -2$,
(5) $m = 2$, $\kappa(X) = 1$, $K^2 = -1$,
(6) $m = 2$, $\kappa(X) = 2$, $K^2 = 0$,
(7) $m_1 = m_2 = 2$, $\kappa(X) = 1$, $K^2 = -2$,
(8) $m_1 = m_2 = 2$, $\kappa(X) = 2$, $K^2 = -1$.

If X is ruled we can use the same arguments as in (3.4.2). We have $q \leq 1$ and by [I, (1.4)] $K^2 \leq -1$. By [F_7, (5.13), (5.15)] we have $s \leq 6$ if $q = 1$; $s \leq 14$ if $q = 0$. Moreover $m = 2$ if $K^2 = -6, -5, -4$; $m = 2, 3$ if $K^2 = -3$; $m \leq 5$ if $K^2 = -2$ and $m \leq 10$ if $K^2 = -1$.

If $2K + L$ is not nef and $2K_0 + L_0$ is nef then $r \leq 15$. If $r \geq 6$ then X is ruled and we can do the same study as before for each value of r.

The list of polarized surfaces with $\Delta = 3$, $d = 5$, and $g \leq \Delta$ can be found in Table III.

4. The case $d = 4$

Let $g \leq \Delta = 3$. By (0.2) we have $g = 2, 3$.

(4.1) $g = 2$. By [F_6, BLP] and (0.6) we see that either (X, L) is a scroll with $L \equiv C_0 + \frac{1}{2}(4+e)f$, $e = -2, 0, 2$, or (X, L) is a $\mathbf{P}^1$-bundle with $q = 1$, $L \equiv 2C_0 + (e+1)f$, and $e = -1, 0$.

(4.2) $g = 3$. By the adjunction formula we have $K \cdot L = 0$. Hence either $\kappa(X) = 0$ or $\kappa(X) = -\infty$. In the first case X is a minimal model, $h^i(L) = 0$ for $i > 0$ and hence

$$\chi(L) = h^0(L) = \chi(\vartheta_X) + \tfrac{1}{2}L \cdot (L - K) = \chi(\vartheta_X) + 2.$$

Thus $\chi(\vartheta_X) = 1$ and X is an Enriques surface.

In the second case, $X \neq \mathbf{P}^2$. By (0.7) we know that $K + L$ is nef unless (X, L) is a scroll. Let (X, L) be a scroll. Then as in (1.3.1) we see that $L \equiv C_0 + \frac{1}{2}(4+e)f$, with $e = -2, 0, 2$.

From now on we assume that $K + L$ is nef. Since X is ruled, by (1.1.0) $q \leq 1$. Moreover by (1.1) X is not a $\mathbf{P}^1$-bundle. Thus there exists a smooth rational curve E on X such that $E^2 = -1$.

We first look at the case $2K + L$ nef. By (3.2.1) $0 \leq 4K^2 + 4$. Hence $K^2 \geq -1$. Moreover since $K \cdot L = 0$, by the Hodge Index Theorem we have $K^2 < 0$. Hence $K^2 = -1$.

If $q = 1$ then X is obtained by blowing up one point on a surface X' with (X', L') a $\mathbf{P}^1$-bundle. Let E be an exceptional curve E on X. Since $2K+L$ is nef we have $L \cdot E = m \geq 2$. Hence $L \equiv aC_0 + bf - mE$ and $K \equiv -2C_0 - ef + E$. Therefore $4 = L^2 = -a^2e + 2ab - m^2$ and $0 = K \cdot L = ae - 2b + m$. Hence $2b = m + ae$ and $4 = am - m^2 = m(a - m)$. We may assume $a - m \geq m$ (see [F_6, (4.10)]). Hence $m = 2$, $a = 4$. Note that if $e < 0$ then $e \geq -q = -1$, and if $e \geq 0$ then from the ampleness of L and from $2b = m + ae$ we get that $e = 0$. Thus either $L \equiv 4C_0 - f - 2E$, $e = -1$, or $L \equiv 4C_0 + f - 2E$, $e = 0$.

If $q = 0$ and $K^2 = -1$ then X is obtained by blowing up either nine points on $\mathbf{F}_e$ or ten points on $\mathbf{P}^2$. By [LP, Theorem 2.5] it is easy to see that $K + L$ is ample. Moreover $(K + L)^2 = 3$ and $g(K + L) = 2$. Thus by the classification of surfaces of sectional genus two the pair (X, L) can be as in one of the following cases:

$$(4.2.1) \quad \begin{cases} \left(\mathbf{F}_{0_9}, 4C_0 + 5f - 2\sum_1^9 E_i\right), & \left(\mathbf{F}_{0_9}, 5C_0 + 4f - 2\sum_1^9 E_i\right), \\ \left(\mathbf{F}_{1_9}, 4C_0 + 7f - 2\sum_1^9 E_i\right), & \left(\mathbf{F}_{2_9}, 4C_0 + 5f - 2\sum_1^9 E_i\right). \end{cases}$$

We now look at the case $2K + L$ not nef. By (0.2) there is an exceptional line E. Let (X', L') be the pair obtained from (X, L) after we blow down E. Note that $d' = L'^2 = 5$, $g' = g = 3$, and $K' \cdot L' = -1$. Hence either our pair (X', L') is as in (b) or there exists a smooth rational curve E on X such that $E^2 = -1$.

If (X', L') is as in (b) then we have $q = 1$, $(X', L') = (\mathbf{P}^1\text{-bundle}, 5C_0 - 2f)$, and $e = -1$.

If (X', L') has an exceptional curve E then we look at Table III (the case $d = 5$, $g = 3$ and $2K + L$ nef or $2K + L$ not nef), since the argument used does not depend on the assumption $\Delta = 3$.

The case $g > \Delta = 3$ requires a completely different technique. We only note that if $Bs|L| = \varnothing$ then $\Phi\colon X \to \mathbf{P}^2$ is a 4-tuple cover of $\mathbf{P}^2$, where Φ is the morphism associated to the linear system $|L|$. In fact $\deg \Phi = 1$ or 4 and in the first case $g(L) \leq 3$ which contradicts $g > \Delta = 3$.

The list of polarized surfaces with $\Delta = 3$, $d = 4$, and $g \leq \Delta$ can be found in Table IV.

For $d = 3, 2$ we study only the case $g \leq \Delta = 3$.

5. The case $d = 3$

Let $g \leq \Delta = 3$. By (0.5) we have $g = 2, 3$.

(5.1) $g = 2$. By [F_6, BLP] and (0.6) we see that either (X, L) is a scroll over a curve of genus two with $L \equiv C_0 + \frac{1}{2}(3 + e)f$, $e = -1, 1$, or (X, L) is an elliptic $\mathbf{P}^1$-bundle with $L \equiv 3C_0 - f$, $e = -1$.

(5.2) $g = 3$. By the adjunction formula we have $K \cdot L = 1$. Hence by the Hodge Index Theorem $K^2 \leq 0$. We also know by (0.7) that $K + L$ is nef unless (X, L) is a scroll. Assume that (X, L) is a scroll. Then as in (2.1) we see that $L \equiv C_0 + \frac{1}{2}(3 + e)f$, with $e = -3, -1, 1$.

From now on we assume that $K + L$ is nef. We will consider separately the case K nef and K not nef.

(5.2.1) K nef. Since K is nef, $K^2 \geq 0$ and X has to be a minimal model. But $K^2 \leq 0$ by the Hodge Index Theorem. Thus $K^2 = 0$. Since $K \cdot L = 1$ we easily see that $\kappa(X) = 1$.

(5.2.2) K not nef. Note that $X \neq \mathbf{P}^2$. Moreover by (1.1) X is not a $\mathbf{P}^1$-bundle. Thus there exists a smooth rational curve E on X such that $E^2 = -1$.

We look first at the case $2K + L$ nef. If $2K + L$ is nef then by (3.2.1) $0 \leq 4K^2 + 7$. Hence $K^2 \geq -1$ and thus $K^2 = -1, 0$. As in [F_6, (4.9)] we conclude that X is ruled and by (1.1.0) $q \leq 1$. Moreover since $L \cdot K = 1$ then by [I, (1.4)] $K^2 \leq -1$.

Let $K^2 = -1$ and $q = 1$. Then X is obtained by blowing up one point on a $\mathbf{P}^1$-bundle. Let $L \equiv aC_0 + bf - mE$, where $L \cdot E = m \geq 2$, and E an exceptional curve on X. Thus

$$3 = L^2 = -a^2 e + 2ab - m^2 \quad \text{and} \quad 1 = K \cdot L = ae - 2b + m.$$

Hence $2b = m + ae - 1$ and $m^2 - am + a + 3 = 0$. From the latter equation we have $(2m - a)^2 = a^2 - 4a - 12$. Completing the squares we have $(a - 2)^2 - (2m - a)^2 = 16$, i.e. $(a - m - 1)(m - 1) = 4$. This last equality along with [F_7, (5.15)] gives either $a = 7$ and $m = 2$, or $a = 6$ and $m = 3$. Note that $e \geq -1$. Thus for $e = -1$, from $2b = m + ae - 1$ we have $b = -3$ if $a = 7$ and $m = 2$, and $b = -2$ if $a = 6$ and $m = 3$. If $e \geq 0$ then by the ampleness of L we have $e = 0$. Hence for $e = 0$ we have $b = 1$ if $a = 6$ and $m = 3$. Hence, concluding, $L \equiv 7C_0 - 3f - 2E$, $e = -1$, or $L \equiv 6C_0 + (3e+1)f - 3E$, $e = -1, 0$.

Let $K^2 = -1$ and $q = 0$. Then X is obtained by blowing up either nine points on $\mathbf{F}_e$ or ten points on $\mathbf{P}^2$. By [LP, Theorem 2.5] it is easy to see that $K+L$ is ample. Moreover $(K+L)^2 = 4$ and $g(K+L) = h^0(K+L) = 3$. Thus $\Delta(X, K+L) = 3$. By our previous classification the pair (X, L) can be one of the following:

$$(5.2.3)\quad \begin{cases} \left(\mathbf{F}_{0_9}, 6C_0 + 7f - 3\sum_1^9 E_i\right), \text{ or } \left(\mathbf{F}_{0_9}, 7C_0 + 6f - 3\sum_1^9 E_i\right), \text{ or} \\ \left(\mathbf{F}_{1_9}, 6C_0 + 10f - 3\sum_1^9 E_i\right), \text{ or } \left(\mathbf{F}_{2_9}, 6C_0 + 9f - 3\sum_1^9 E_i\right), \text{ or} \\ \left(\mathbf{P}^2_{10}, \vartheta_{\mathbf{P}^2}(12) - 4\sum_1^8 E_i - 3E_9 - 2E_{10}\right) \end{cases}$$

Let $2K+L$ be not nef. We write (X', L') for the pair obtained from (X, L) after we blow down the exceptional line E. Note that $d' = L'^2 = 4$, $g' = g = 3$, and $K' \cdot L' = 0$. Hence either $\kappa(X) = 0$ or $\kappa(X) = -\infty$. If $\kappa(X) = 0$ then we conclude that the surface X' is either Enriques, or abelian, or bielliptic. Let $\kappa(X) = -\infty$. X' cannot be a $\mathbf{P}^1$-bundle. Hence there exists an exceptional curve E on X'. Thus we have to look at Table IV (the case $d = 4$, $g = 3$, and $2K + L$ nef or $2K + L$ not nef).

The list of polarized surfaces with $\Delta = 3$, $d = 3$, and $g \leq \Delta$ can be found in Table V.

6. The case $d = 2$

Let $g \leq \Delta = 3$. By (0.5) we have $g = 2, 3$.

(6.1) $g = 2$. By [F_6, BLP] and (0.6) we see that either (X, L) is a scroll with $L \equiv C_0 + \frac{1}{2}(2+e)f$, $e = -2, 0$, or $q = 1$ and (X, L) is obtained by blowing up either one point on a $\mathbf{P}^1$-bundle with $e = -1$, or blowing up two points on a $\mathbf{P}^1$-bundle with $e = -1, 0$. Note that $L \equiv 3C_0 - f - E$ in the first case and $L \equiv 2C_0 + (e+1)f - E_1 - E_2$ in the second case; or (X, L) is abelian or bielliptic.

(6.2) $g = 3$. By the adjunction formula we have $K \cdot L = 2$. Hence by the Hodge Index Theorem $K^2 \leq 2$. We also know that $K + L$ is nef unless

(X, L) is a scroll. Assume that (X, L) is a scroll. Then as in (2.1) we see that $L \equiv C_0 + \frac{1}{2}(2+e)f$, with $e = -2, 0$.

From now on we assume that $K + L$ is nef.

(6.2.1) K *nef*. In such a case we have $K^2 \geq 0$ and X is a minimal model. But $K^2 \leq 2$ by the Hodge Index Theorem. Thus $0 \leq K^2 \leq 2$. Since $K \cdot L = 2$ we easily see that $k(X) = 2$ if $K^2 = 1, 2$ and $k(X) = 1$ if $K^2 = 0$.

(6.2.2) K *not nef*. Note that $X \neq \mathbf{P}^2$ and by (1.1) X is not a $\mathbf{P}^1$-bundle. Thus there exists a smooth rational curve E on X such that $E^2 = -1$.

We look first at the case $2K + L$ nef. If $2K + L$ is nef then by (3.2.1) $0 \leq 4K^2 + 10$. Hence $K^2 \geq -2$ and thus $-2 \leq K^2 \leq 2$. As in [F_6, (4.9)] we see that either X is ruled or $\kappa(X) = 0$. In the latter case X is obtained by blowing up one point with $m = 2$. Hence $K^2 = -1$. Since $d' = 6$ and $g(L') = 4$ then X is gotten by blowing up one double point on a K-3 surface or on a surface of type Σ, see §2. If X is ruled then by (1.1.0) $q \leq 1$ and by [I, (1.4)], $K^2 \leq -1$ since $L \cdot K = 2$.

Let $q = 1$. If $K^2 = -1$ then X is obtained by blowing up one point on X', with X' a $\mathbf{P}^1$-bundle. Let $L \equiv aC_0 + bf - mE$, where $L \cdot E = m \geq 2$, and E an exceptional curve on X. Thus

$$2 = L^2 = -a^2e + 2ab - m^2 \quad \text{and} \quad 2 = K \cdot L = ae - 2b + m.$$

Hence $2b = m + ae - 2$ and $m^2 - am + 2a + 2 = 0$. From the latter equation we have $(2m - a)^2 = a^2 - 8a - 8$. Completing the squares we have $(a-4)^2 - (2m-a)^2 = 24$, i.e. $(a - m - 2)(m - 2) = 6$. This last equality along with [F_7, (5.15)] gives either $a = 11$ and $m = 3$, or $a = 9$ and $m = 4$. Note that $e \geq -1$. Thus for $e = -1$, from $2b = m + ae - 2$ we have $b = -5$, $a = 11$ and $m = 3$. If $e \geq 0$ then by the ampleness of L we have $e = 0$. We have $b = 1$, $a = 9$ and $m = 4$. Hence, concluding, $L \equiv 11C_0 - 5f - 3E$, $e = -1$, or $L \equiv 9C_0 + f - 4E$, $e = 0$. If $K^2 = -2$ then X is obtained by blowing up two points on X', with X' a $\mathbf{P}^1$-bundle. As before we have $L \equiv 5C_0 + f - 2Q_1 - 2Q_2$ and $e = 0$.

Let $q = 0$. In the case $K^2 = -2$, by [F_7, (5.15)] $m = 2$. Moreover since $g(K + L) = 3$, $(K + L)^2 = 4$, and $\Delta(X, K + L) = 3$, using our previous classification we see that this case does not occur. If $K^2 = -1$ then $m \leq 4$ and X is obtained by blowing up 9 points on $\mathbf{F}_e$.

If $2K + L$ is not nef then we let (X', L') be the pair obtained from (X, L) after we blow down the exceptional line E. Note that $d' = L'^2 = 3 = g' = g$ and $K' \cdot L' = 1$. By the Hodge Index Theorem $K'^2 \leq 0$. With the usual arguments we see that $\kappa(X) \neq 0, 2$ since $K' \cdot L' = 1$ and $K'^2 \leq 0$. If $\kappa(X) = 1$ then $r = 1$, $K^2 = -1$. We only have to look at the case $\kappa(X) = -\infty$. X' cannot be a $\mathbf{P}^1$-bundle. Thus there exists an exceptional line E on X'. Hence we have to look at Table V (the case $d = 3 = g$ and $2K + L$ nef or $2K + L$ not nef).

The list of polarized surfaces with $\Delta = 3$, $d = 2$, and $g \leq \Delta$ can be found in Table VI.

In Tables I–VI $\Delta = 3$, (X_0, L_0) denotes the relatively half-minimal model of (X, L), and r denotes the number of points that we have to blow up on (X_0, L_0) to get (X, L).

TABLE I. $d \geq 7$

g	$k(X)$	d	(X, L)	q	$K \cdot K$	(X_0, L_0)	r
2	$-\infty$	≥ 7	$(\text{Scroll}, C_0 + \frac{1}{2}(d+e)f)$, $e = 1, \dots, d-2$; if $e = -1$ then $d = 7$	2			
3		≥ 7	$(\text{Scroll}, C_0 + \frac{1}{2}(d+e)f)$, $e = 0, \dots, d-2$; if $e = -3, -1$ then $d = 7$	3			
		$16, \dots, 7$		0	$8, \dots, 0$	$(\mathbf{F}_e, 2C_0 + (e+4)f)$, $e \leq 3$	$0, \dots, 9$
		$16, \dots, 7$		0	$9, \dots, 1$	$(\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(4))$	$0, \dots, 9$
		$8, 7$		0	$2, 1$	$\left(\mathbf{P}^2_7, \vartheta_{\mathbf{P}^2}(6) - 2\sum_{i=1}^{7} E_i\right)$	$0, 1$

TABLE II. $d = 6$

g	$\kappa(X)$	$K + L$	(X, L)	q	$K \cdot K$	(X_0, L_0)	r
2	$-\infty$	not nef	$(\text{Scroll}, C_0 + \frac{1}{2}(6+e)f)$ $e = 2, 4$	2			
3			$(\text{Scroll}, C_0 + \frac{1}{2}(6+e)f)$ $e = -2, 0, 2$	3			
		nef and $2K + L$ not nef		0	0	$\left(\mathbf{P}^2_7, \vartheta_{\mathbf{P}^2}(6) - 2\sum_{i=1}^{7} E_i\right)$	2
				1	-2	$(\mathbf{P}^1\text{-bundle}, 2C_0 + (2+e)f)$, $e = -1, 0, 1$	
				0	-1	$(\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(4))$	10
				0	-2	$(\mathbf{F}_e, 2C_0 + (4+e)f)$, $e \leq 3$	
4	0	nef	K-3 surface, type Σ	0	0		
≥ 5			type Σ				

TABLE III. $d = 5$, $g \leq \Delta$

g	$\kappa(X)$	$K + L$	(X, L)	q	$K \cdot K$	(X_0, L_0)	r
2	$-\infty$	not nef	$(\text{Scroll}, C_0 + \frac{1}{2}(5+e)f)$, $e = -1, 1, 3$	2			
3			$(\text{Scroll}, C_0 + \frac{1}{2}(5+e)f)$, $e = -3, -1, 1, 3$	3			
		nef and $2K + L$ nef	$\left(\mathbf{P}^2_9, \vartheta_{\mathbf{P}^2}(9) - 3\sum_{i=1}^{8} E_i - 2E_9\right)$	0	0		
		nef and $2K + L$ not nef		1	-1	$(\mathbf{P}^1\text{-bundle}, 3C_0 + f)$, $e = 0$	1
				1	-3	$(\mathbf{P}^1\text{-bundle}, 2C_0 + (2+e)f)$, $e = -1, 0, 1$	3
				0	-1	$\left(\mathbf{P}^2_7, \vartheta_{\mathbf{P}^2}(6) - 2\sum_{i=1}^{7} E_i\right)$	
				0	-2	$(\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(4))$	11
				0	-3	$(\mathbf{F}_e, 2C_0 + (4+e)f)$, $e \leq 3$	

TABLE IV. $d = 4,\ g \leq \Delta$

g	$\kappa(X)$	$K+L$	(X, L)	q	$K \cdot K$	(X_0, L_0)	r
2	$-\infty$	not nef	(Scroll, $C_0 + \frac{1}{2}(4+e)f$), $e = -2, 0, 2$	2			
		nef	($\mathbf{P}^1$-bundle, $2C_0 + (e+1)f$), $e = -1, 0$	1	0		
3		not nef	(Scroll, $C_0 + \frac{1}{2}(4+e)f$), $e = -2, 0, 2$	3			
		nef and $2K+L$ nef	($\mathbf{P}^1$-bundle)$_1$, $4C_0 + (2e+1)f - 2E$, $e = -1, 0$	1	-1		
			$\mathbf{F}_{e_9}$ (see (4.2.1))	0			
		nef and $2K+L$ not nef		1	0	($\mathbf{P}^1$-bundle, $5C_0 - 2f$), $e = -1$	1
				0	-1	$\left(\mathbf{P}^2_9, \vartheta_{\mathbf{P}^2}(9) - 3\sum_{i=1}^{8} E_i - 2E_9\right)$	
				1	-2	($\mathbf{P}^1$-bundle, $3C_0 + f$), $e = 0$	2
					-4	($\mathbf{P}^1$-bundle, $2C_0 + (2-e)f$), $e = -1, 0, 1$	4
				0	-2	$\left(\mathbf{P}^2_7, \vartheta_{\mathbf{P}^2}(6) - 2\sum_{i=1}^{7} E_i\right)$	
				0	-3	($\mathbf{P}^2$, $\vartheta_{\mathbf{P}^2}(4)$)	12
				0	-4	($\mathbf{F}_e$, $2C_0 + (4+e)f$), $e \leq 3$	
	0	nef	Enriques surface	0	0		

TABLE V. $d = 3,\ g \leq \Delta$

g	$k(X)$	$K+L$	(X, L)	q	$K \cdot K$	(X_0, L_0)	r
2	$-\infty$	not nef	(Scroll, $C_0 + \frac{1}{2}(3+e)f$), $e = -1, 1$	2			
		nef	($\mathbf{P}^1$-bundle, $3C_0 - f$), $e = -1$	1	0		
3		not nef	(Scroll, $C_0 + \frac{1}{2}(3+e)f$), $e = -3, -1, 1$	3			
		nef and $2K+L$ nef	($\mathbf{P}^1$-bundle)$_1$, $7C_0 - 3f - 2E$, $e = -1$ ($6C_0 + (3e+1)f - 3E$), $e = -1, 0$	1	-1		
			$\mathbf{F}_{e_9}$ (see (5.2.3))	0			
		nef and $2K+L$ not nef		1	-2	($\mathbf{P}^1$-bundle)$_1$, $4C_0 + (2e+1)f - 2E$, $e = -1, 0$	1
				0	-2	$\mathbf{F}_{e_9}$ (see (4.2.1))	
				1	-1	($\mathbf{P}^1$-bundle, $5C_0 - 2f$), $e = -1$	2
				0	-2	$\left(\mathbf{P}^2_9, \vartheta_{\mathbf{P}^2}(9) - 3\sum_{i=1}^{8} E_i - 2E_9\right)$	
				1	-3	($\mathbf{P}_1$-bundle, $3C_0 + f$), $e = 0$	3
					-5	($\mathbf{P}^1$-bundle, $2C_0 + (2+e)f$), $e = -1, 0, 1$	5
				0	-3	$\left(\mathbf{P}^2_7, \vartheta_{\mathbf{P}^2}(6) - 2\sum_{i=1}^{7} E_i\right)$	
					-4	($\mathbf{P}^2$, $\vartheta_{\mathbf{P}^2}(4)$)	13
					-5	($\mathbf{F}_e$, $2C_0 + (4+e)f$), $e \leq 3$	
	0				-1	Enriques, abelian, bielliptic	1
	1	nef	minimal elliptic		0		

TABLE VI. $d = 2$, $g \leq \Delta$

g	$k(X)$	$K + L$	(X, L)	q	$K \cdot K$	(X_0, L_0)	r
2	$-\infty$	not nef	(Scroll, $C_0 + \frac{1}{2}(e+2)f$), $e = -2, 0$	2			
		nef	$((\mathbf{P}^1\text{-bundle})_1, 3C_0 - f - E)$, $e = -1$	1	-1		
			$((\mathbf{P}^1\text{-bundle})_2, 2C_0 + (e+1)f - E_1 - E_2)$, $e = -1, 0$		-2		
	0		abelian, bielliptic	2, 1	0		
3	$-\infty$	not nef	(Scroll, $C_0 + \frac{1}{2}(e+2)f$), $e = -2, 0$	3			
		nef and $2K + L$ nef	$(\mathbf{P}^1\text{-bundle})_1$, $11C_0 - 5f - 3E$, $e = -1$, $9C_0 + f - 4E$, $e = 0$	1	-1		
			$(\mathbf{P}^1\text{-bundle})_2$, $5C_0 + f - 2Q_1 - 2Q_2$, $e = 0$		-2		
			$\mathbf{F}_{e_9}$	0	-1		
	0		K-3, type Σ with one double point blown up	0	-1		
	$-\infty$	nef and $2K + L$ not nef		1	-1	$(\mathbf{P}^1\text{-bundle})_1$, $7C_0 - 3f - 2E$, $e = -1$, $6C_0 + (3e+1)f - 3E$, $e = -1, 0$	1
				0		$\mathbf{F}_{e_9}$ (see (5.2.3))	2
				1	-3	$(\mathbf{P}^1\text{-bundle})_1$, $4C_0 + (2e+1)f - 2E$, $e = -1, 0$	
				0	-3	$\mathbf{F}_{e_9}$ (see (4.2.1))	
				1	-2	$(\mathbf{P}^1\text{-bundle}, 5C_0 - 2f)$, $e = -1$	3
				0	-3	$\left(\mathbf{P}^2_9, \vartheta_{\mathbf{P}^2}(9) - 3\sum_{i=1}^{8} E_i - 2E_9\right)$	
				1	-4	$(\mathbf{P}^1\text{-bundle}, 3C_0 + f)$, $e = 0$	4
					-6	$(\mathbf{P}^1\text{-bundle}, 2C_0 + (2+e)f)$, $e = -1, 0, 1$	6
				0	-4	$\left(\mathbf{P}^2_7, \vartheta_{\mathbf{P}^2}(6) - 2\sum_{i=1}^{7} E_i\right)$	
					-5	$(\mathbf{P}^2, \vartheta_{\mathbf{P}^2}(4))$	14
					-6	$(\mathbf{F}_e, 2C_0 + (4+e)f)$, $e \leq 3$	
	1	nef	minimal elliptic		0		
	2		minimal general type		1, 2		
	1				-1	minimal elliptic	1

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